- Math 72 9.4

Math 62 11.4

Properties of Logarithms
and the change-of base
formula

Math 70

More Properties of Logarithms

Objectives

"combine"-type problems

1) Combine $\log_b a + \log_b c$ using property
2) Combine $\log_b a - \log_b c$ using property
3) Repeatedly combine $\log_b a + \log_b a + \ldots = k \log_b a$
4) Use these properties together to combine to a single log with coefficient 1.

"separate"-type problems.

5) Write a single log as a sum, difference and/or multiple of simpler logs using the same properties, in the opposite direction.
6) Given numerical values for a few simple logs, separate a more complicated log into those simpler logs to find its numerical value.

7) Use the change-of-base formula
   - to calculate $\log_b(a)$ where $b \neq 10$ or e
   - to graph $y = \log_b(x)$ where $b \neq 10$ or e

Math 70 (More) Properties of Logarithms, Day 1

Consider the sum of two logs, same base, b:

$$\log_b x + \log_b y$$

Let's make a substitution to make this simpler.

$$M = \log_b x \longleftrightarrow b^M = x$$
$$N = \log_b y \longleftrightarrow b^N = y$$

$$= M + N$$

Using the inverse property:

$$= \log_b b^{M+N}$$

Rules of exponents tell us:

$$= \log_b (b^M \cdot b^N)$$

Substitute back:

$$= \log_b (x \cdot y)$$

$$\boxed{\text{Beginning} = \text{End}: \quad \log_b x + \log_b y = \log_b xy}$$

← Sometimes called the Product property of logs.

Similarly:

$$\log_b x - \log_b y$$
$$= M - N$$
$$= \log_b b^{M-N}$$
$$= \log_b \left(\frac{b^M}{b^N}\right)$$
$$= \log_b \left(\frac{x}{y}\right).$$

Giving

$$\boxed{\log_b x - \log_b y = \log_b \left(\frac{x}{y}\right)}$$

← Sometimes called the Quotient Property of Logs.

Remember $x+x+x = 3x$ multiplication is repeated addition.

So $\log_b x + \log_b x + \log_b x = 3\log_b x$.

Using the same methods as before:

$\log_b x = M$ means $b^M = x$.

$3\log_b x$

$= 3M$ Subst

$= \log_b b^{3M}$ Inverse property.

$= \log_b((b^M)^3)$ exponent rules.

$= \log_b(x^3)$. subst back ←

This means

$$3\log_b x = \log_b x^3$$

We can expand this for any multiple k:

$$\boxed{k\cdot\log_b x = \log_b x^k}$$

← This is sometimes called the Power Rule for exponents.

Write each sum as a single log.

① $\log_{11}10 + \log_{11}3 = \log_{11}(10\cdot3) = \boxed{\log_{11}33}$

② $\log_3\frac{1}{2} + \log_3 12 = \log_3(\frac{1}{2}\cdot12) = \boxed{\log_3 6}$

③ $\log_2(x+2) + \log_2 x = \log_2[x\cdot(x+2)] = \boxed{\log_2(x^2+2x)}$

Write each difference as a single log.

④ $\log_{10} 27 - \log_{10} 3 = \log_{10}\left(\frac{27}{3}\right) = \boxed{\log_{10}(9)}$

⑤ $\log_5 8 - \log_5 x = \boxed{\log_5 \frac{8}{x}}$

⑥ $\log_3 (x^2+5) - \log_3 (x^2+1) = \boxed{\log_3 \frac{x^2+5}{x^2+1}}$

Write each multiple as a single log.

⑦ $2 \log_7 3 = \log_7 3^2 = \boxed{\log_7 9}$

⑧ $4 \log_3 x = \boxed{\log_3 x^4}$

⑨ $-5 \log_9 2 = \log_9 2^{-5} = \boxed{\log_9 \frac{1}{32}}$

In MathXL, do all 9. problems asking you to write as a single log.

⑩ $2 \log_5 3 + 3 \log_5 2 = \log_5 3^2 + \log_5 2^3 = \log_5 9 \cdot 8 = \boxed{\log_5 72}$

⑪ $3 \log_9 x - \log_9 (x+1) = \log_9 x^3 - \log_9 (x+1) = \boxed{\log_9 \frac{x^3}{x+1}}$

⑫ $\log_4 25 + \log_4 3 - \log_4 5 = \log_4\left(\frac{25 \cdot 3}{5}\right) = \boxed{\log_4 15}$

Extras

⑬ $5 \log_6 x - \frac{3}{4} \log_6 x + 3 \log_6 x = \log_6 x^5 - \log_6 x^{3/4} + \log_6 x^3 = \log_6 \frac{x^5 \cdot x^3}{x^{3/4}}$

$= \log_6 \frac{x^8}{x^{3/4}}$

$= \boxed{\log_6 x^{29/4}}$

⑭ $2\log_5 x + \frac{1}{3}\log_5 x - 3\log_5(x+5)$

$= \log_5 x^2 + \log_5 x^{1/3} - \log_5 (x+5)^3$

$= \log_5 \frac{x^2 \cdot x^{1/3}}{(x+5)^3}$

add exponents

$2 + \frac{1}{3}$

$= \frac{6}{3} + \frac{1}{3}$

$= \frac{7}{3}$

$= \boxed{\log_5 \frac{x^{7/3}}{(x+5)^3}}$

⑮ $2\log_7 y + 6\log_7 z$

$= \log_7 y^2 + \log_7 z^6$

$= \boxed{\log_z (y^2 z^6)}$

SUMMARY OF LOG PROPERTIES $b \neq 1,\ b > 0$

1) $\log_b 1 = 0$

2) $\log_b b^x = x$

3) $b^{\log_b x} = x$

4) $\log_b x \cdot y = \log_b x + \log_b y$

5) $\log_b \frac{x}{y} = \log_b x - \log_b y$

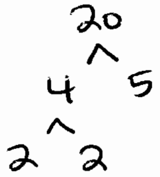
6) $\log_b x^k = k \cdot \log_b x$

Math 70 "Separate"-type problems

Write each log as a sum of logs.

① $\log_3(20)$

<u>step 1</u>: Factor 20 to prime factors



$20 = 2^2 \cdot 5$

$\log_3(20) = \log_3(2^2 \cdot 5)$

<u>step 2</u>: use properties 1 & 2 first

$= \log_3(2^2) + \log_3(5)$

<u>step 3</u>: use property 3 last

$= \boxed{2\log_3(2) + \log_3(5)}$

or $\boxed{2\log_3 2 + \log_3 5}$

② $\log_2(4y) = \log_2(2^2 \cdot y)$

$= \log_2(2^2) + \log_2 y$ ← remember inverse functions! $\log_2(2^x) = x$

$= \boxed{2 + \log_2 y}$

③ $\log_5(ab) = \boxed{\log_5 a + \log_5 b}$

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Write each log as a difference of logs

④ $\log_3\left(\frac{5}{4}\right) = \log_3(5) - \log_3(4)$

$= \log_3(5) - \log_3(2^2)$

$= \boxed{\log_3 5 - 2\log_3 2}$

⑤ $\log_2\left(\frac{8}{y}\right) = \log_2(8) - \log_2(y)$

$= \log_2(2^3) - \log_2(y)$

$= \boxed{3 - \log_2 y}$

inverses again!

$\log_2 2^x = x$

⑥ $\log_5\left(\frac{a}{b}\right) = \boxed{\log_5 a - \log_5 b}$

Write each log as the multiple of a log

⑦ $\log_3 x^5 = \boxed{5 \cdot \log_3 x}$ or $\boxed{5\log_3 x}$

⑧ $\log_4 \sqrt{x} = \log_4 x^{1/2}$

$= \boxed{\frac{1}{2}\log_4 x}$

remember $\sqrt{x} = x^{1/2}$

⑨ $\log_7(a^b) = \boxed{b \cdot \log_7 a}$

Write each log as the sum and/or difference of multiples of logs.

⑩ $\log_3\left(\frac{35}{44}\right) = \log_3\left(\frac{5 \cdot 7}{2^2 \cdot 11}\right)$

find prime factors

$= \log_3 5 + \log_3 7 - \log_3 2^2 - \log_3 11$

$= \boxed{\log_3 5 + \log_3 7 - 2\log_3 2 - \log_3 11}$

factors in numerator are added; factors in denom are subtracted

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⑪ $\log_2 \frac{x^5}{(y+1)^2} = \log_2 x^5 - \log_2 (y+1)^2$

$$= \boxed{5 \log_2 x - 2 \log_2 (y+1)}$$

↑

Note: parentheses are required.
$\log_2 y+1$ means $\log_2(y) + 1$
or $\log_2(y) + \log_2(2)$

⑫ $\log_3 \left(\frac{x+5}{x}\right)^2$

option 1: outside exponent first

$$= 2 \log_3 \left(\frac{x+5}{x}\right)$$

$$= 2\left[\log_3 (x+5) - \log_3 (x)\right]$$

$$= \boxed{2 \log_3 (x+5) - 2 \log_3 (x)}$$

option 2: use exponent properties first

$$= \log_3 \frac{(x+5)^2}{x^2}$$

$$= \log_3 (x+5)^2 - \log_3 x^2$$

$$= \boxed{2 \log_3 (x+5) - 2 \log_3 (x)}$$

↑

parentheses required

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Given $\log_b 2 = 0.43$

and $\log_b 3 = 0.68$,

Use properties of logs
to evaluate the following:

← Notice: base
not given.
Don't know
any logs excep
$\log_b 2$ and $\log_b 3$

⑬ $\log_b 6$

step 1: Write argument in prime factors and powers

$6 = 2 \cdot 3$

step 2: Rewrite log using prime factors

$= \log_b 2 \cdot 3$

step 3: Write as sum/difference of multiples of logs.

$= \log_b 2 + \log_b 3$

step 4: Substitute the given values

$$= \underbrace{\log_b 2}_{\downarrow} + \underbrace{\log_b 3}_{\downarrow}$$

$$= 0.43 \qquad 0.68$$

step 5: Do arithmetic (must be easy — if you get a mile-long decimal, you made a mistake!)

$= \boxed{1.11}$

Math 70

⑭ $\log_b 9 = \log_b 3\cdot 3$ or $\log_b 3^2$

↙

$= \log_b 3 + \log_b 3$

$= .68 + .68$

$= \boxed{1.36}$

↘

$= 2\log_b 3$

$= 2(0.68)$

$= \boxed{1.36}$

⑮ $\log_b \sqrt{2}$

$= \log_b 2^{1/2}$

$= \frac{1}{2}\log_b 2$

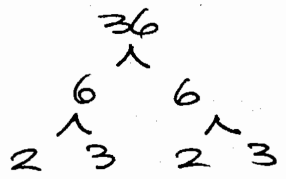
$= \frac{1}{2}(.43)$

$= \boxed{0.215}$

Math 70

same instructions

⑯ $\log_b 36$



$36 = 2^2 \cdot 3^2$

$= \log_b(2^2 \cdot 3^2)$

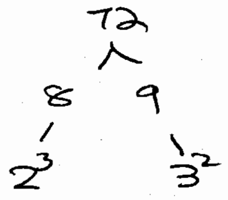
$= \log_b 2^2 + \log_b 3^2$

$= 2\log_b 2 + 2\log_b 3$

$= 2(0.43) + 2(0.68)$

$= \boxed{2.22}$

⑰ $\log_b 72$



$= \log_b(2^3 \cdot 3^2)$

$= \log_b 2^3 + \log_b 3^2$

$= 3\log_b 2 + 2\log_b 3$

$= 3(0.43) + 2(0.68)$

$= \boxed{2.65}$

⑱ $\log_b \frac{4}{9}$

$= \log_b \frac{2^2}{3^2}$

$= \log_b 2^2 - \log_b 3^2$

$= 2\log_b 2 - 2\log_b 3$

$= 2(.43) - 2(.68)$

$= \boxed{-0.5}$

Math 70

same instructions, continued.

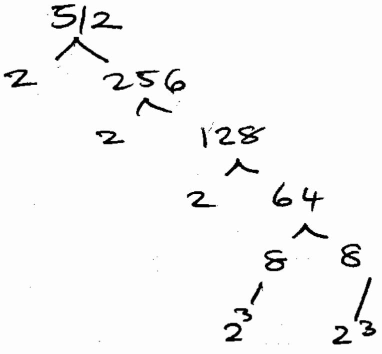
⑲ $\log_b 512$

$= \log_b 2^9$

$= 9\log_b 2$

$= 9(.43)$

$= \boxed{3.87}$



Challenge Problem

⑳ $\log_b \sqrt[3]{1536}$

$= \log_b (1536)^{1/3}$

$= \frac{1}{3}\log_b 1536$

$= \frac{1}{3}\log_b(2^9 \cdot 3)$

$= \frac{1}{3}\log_b 2^9 + \frac{1}{3}\log_b 3$

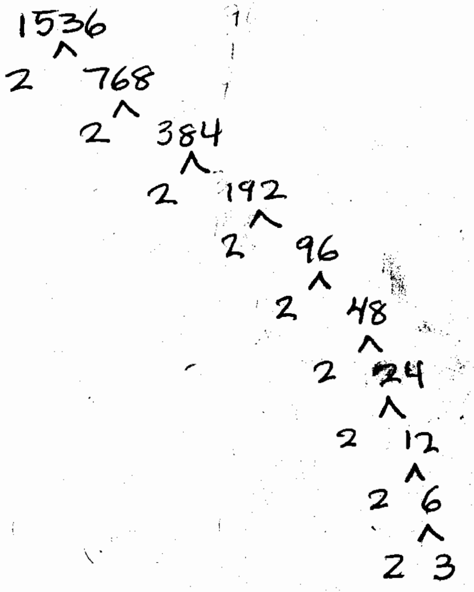
$= 9\cdot\frac{1}{3}\cdot\log_b 2 + \frac{1}{3}\log_b 3$

$= 3\log_b 2 + \frac{1}{3}\log_b 3$

$= 3(.43) + \frac{1}{3}(.68)$

$= \boxed{1.51\overline{6}}$ ← DO NOT ROUND!

$= \boxed{\frac{91}{60}}$



Math 70 MG 5/e 9.6 Day 2

Review:

Write as a single log

⑳ $\log_6 18 + 3\log_6 2 - \log_6 9$

$= \log_6 18 + \log_6 2^3 - \log_6 9$

$= \log_6 18 + \log_6 8 - \log_6 9$

$= \log_6 \left(\frac{18 \cdot 8}{9}\right)$

$= \boxed{\log_6 16}$

㉑ $\log_9 4x - \log_9 (x-3) + \log_9 (x^3+1)$

$= \boxed{\log_9 \frac{4x\,(x^3+1)}{x-3}}$

Write as sum or difference or multiple logs

㉒ $\log_6 \frac{x^2}{x+3}$

$= \log x^2 - \log_6 (x+3)$

$= \boxed{2\log x - \log_6 (x+3)}$

㉓ $\log_5 x^3(x+1)$

$= \log_5 x^3 + \log_5 (x+1)$

$= \boxed{3\log_5 x + \log_5 (x+1)}$

㉔ $\log_2 \frac{x^3}{\sqrt{y}}$

$= \log_2 x^3 - \log_2 \sqrt{y}$

$= \boxed{3\log_2 x - \frac{1}{2}\log_2 y}$

[Technically $x = \log_2 3$ is "Solved". That's the exact answer.]

⑫ Approximate $\log_2 3 = x$ to 4 decimal places.

step 1: Write equivalent exponential equation

$$2^x = 3$$

step 2: Take common logs of both sides of equation.

$$\log 2^x = \log 3$$

step 3: Use property of logs on LHS

$$x \cdot \log 2 = \log 3$$

NOTE: log 2 is a number — a constant! So is log 3! So we can isolate x by dividing both sides by that number

$$\frac{x \cdot \log 2}{\log 2} = \frac{\log 3}{\log 2}$$

CAUTION $\frac{\log 3}{\log 2} \neq \log \frac{3}{2}$

Since $\log \frac{3}{2} = \log 3 - \log 2$

$$\log_2 3 = x = \frac{\log 3}{\log 2}$$ exact

1.58496

$\approx \boxed{1.5850}$ approx

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In problem ⑫, we use common logs because they're on the GC. But we could have used natural logs. Does the result change?

⑫ $\log_2 3 = x$ (It's already solved for x!)

$2^x = 3$

$\ln 2^x = \ln 3$

$x \ln 2 = \ln 3$

$\boxed{x = \log_2 3} = \boxed{x = \frac{\ln 3}{\ln 2}}$ exact

exact

1.58496

$= \boxed{1.5850}$ approx

Solve ÷ logs (2 logs)

VS:

The exact expression is the same, but base e. The approximate result is identical.

log prop ÷ argu- ments ÷ [illegible] logs

CAUTION

log properties are different from change of ba

$\log_b a - \log_b c = \log_b\left(\frac{a}{c}\right)$

$\log 3 - \log 2 = \log\left(\frac{3}{2}\right)$

$\log\left(\frac{3}{2}\right) \approx .1761$

If we'd used base 7 we'd get:

$$\log_2 3 = \frac{\log_7 3}{\log_7 2}$$

or any other base of logarithm.

Change of Base Formula

$\log_b x = \frac{\log x}{\log b}$ to change base b to base 10

$\log_b x = \frac{\ln x}{\ln b}$ to change base b to base e

$\log_b x = \frac{\log_c x}{\log_c b}$ to change base b to base c

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Q. What is the difference, if any, between these?

a) $\dfrac{\log 3}{\log 2}$ b) $\log\left(\dfrac{3}{2}\right)$ c) $\dfrac{\log 3}{2}$

A: They are *all* different from each other!

a) $\dfrac{\log 3}{\log 2} = \log_2 3$ change of base formula

$$\approx \underline{\underline{1.58496}}$$

b) $\log\left(\frac{3}{2}\right) = \log(1.5)$ or $\log(3) - \log(2)$

log property
$\log_b\left(\frac{a}{c}\right) = \log_b a - \log_b c$

$$\approx \underline{\underline{0.17609}}$$

c) $$\frac{\log 3}{2} = \underset{\uparrow \text{ arithmetic}}{\frac{\log(3)}{2}} = \underset{\uparrow \text{ coefficient}}{\frac{1}{2}\log(3)} = \underset{\uparrow}{\log 3^{1/2}} = \underset{\uparrow}{\log\sqrt{3}}$$

log property
$k\log_b a = \log_b a^k$

½ exponent means square root.

$$\approx \underline{\underline{0.23856}}$$

Math 70

Find approximate values. Round to nearest ten-thousandth.

✓ ⑬ $\log_5 7 = \dfrac{\log 7}{\log 5} \approx \boxed{1.2091}$

⑭ $\log_2 1 = \boxed{0}$ (log property!)

✓ ⑮ $\log_2 10 = \dfrac{\log 10}{\log 2} = \boxed{3.3219}$ not the same as $\log_{10} 2$!

✓ ⑯ Use GC to look at graph of $y = \log_2 x$.

$Y_1 = \log(x)/\log(2)$

– OR –

$Y_1 = \ln(x)/\ln(2)$.

✓ ⑰ Calculate $\log \sqrt[3]{10}$

a) without GC.

$\log \sqrt[3]{10} = \log_{10} 10^{1/3} = \boxed{\frac{1}{3}}$ inverse property $\log_b b^x = x$

b) with GC using cube root $\log(\sqrt[3]{10})$

$\boxed{\text{LOG}}\ \boxed{\text{MATH}}\ \underset{\uparrow \atop \sqrt[3]{\ }}{4.}\ 10\ )\ )\ \text{MATH}\ \underset{\uparrow \atop \triangleright\text{frac}}{1.}$

$\log(\sqrt[3]{\ }(10)) \triangleright \text{frac}$

c) with GC using $\frac{1}{3}$ power $\log(10^{1/3})$

$\log(10 \wedge (1/3)) \triangleright \text{frac} = \boxed{\frac{1}{3}}$

CAUTION: NOT $\log(10) \wedge (1/3)$ ⇒ order of operations
nested parentheses inside out